1.4 The Matrix equation $A\vec{x} = \vec{b}$

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Fundamental idea: Any system can be understood in full generality by studying the structure of A.

"Now that we have a solid understanding of vectors, we utilize equations of vectors to succinctly describe linear systems. To even greater effect, we will move to equations of matrices which encode in full generality any linear system. By studying this general form, we will understand more about any linear system from the efficient perspective of matrix theory. To begin, we connect linear systems and vectors:"

A motivating question: Given a collection of vectors $\vec{a}_1, \vec{a}_2, \dots, \vec{a}_n$ and a fixed vector $\vec{b}$,

$\boxed{\text{Is } \vec{b} \in \text{span}\{\vec{a}_1, \dots, \vec{a}_n\}\text{?}}$ — or — is $\vec{b}$ a linear combination of $\vec{a}_1, \dots, \vec{a}_n$?

Ex) Is $\vec{b} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$ a linear combination of $\vec{a}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $\vec{a}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$?

"Equivalently, is $\vec{b} \in \text{span}\{\vec{a}_1, \vec{a}_2\}$?" Notice this amounts to determining if there are $x_1, x_2 \in \mathbb{R}$ s.t.

$$x_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$$

This is the vector equation mentioned in the previous section

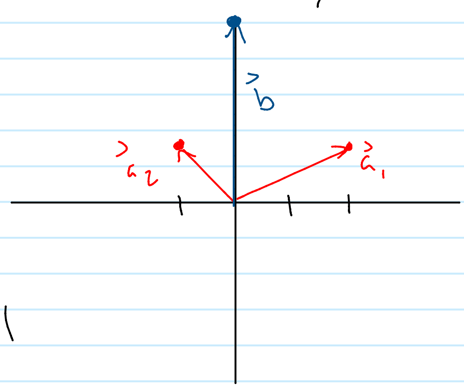
$$\downarrow$$

$$\begin{bmatrix} 2x_1 \\ x_1 \end{bmatrix} + \begin{bmatrix} -x_2 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$$

$$\downarrow$$

$$\begin{bmatrix} 2x_1 - x_2 \\ x_1 + x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$$

Geometrically:



We see that indeed $\vec{a}_1 + 2\vec{a}_2 = \vec{b}$. So $\vec{b}$ is in the part of $\mathbb{R}^2$ spanned by $\vec{a}_1, \vec{a}_2$ (namely all of it).

$$\begin{bmatrix} 2x_1 - x_2 \\ x_1 + x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$$

(a_1, a_2 (namely all of it).

Recall vectors equal if their corresponding entries equal,

so for this equation to be true, need:

$$\begin{aligned} 2x_1 - x_2 &= 0 \\ x_1 + x_2 &= 3 \end{aligned} \;\rightsquigarrow\; \begin{array}{ccc} \vec{a}_1 & \vec{a}_2 & \vec{b} \end{array} \begin{bmatrix} 2 & -1 & 0 \\ 1 & 1 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \end{bmatrix}$$

so $x_1 = 1$, $x_2 = 2$

And yes

$$1 \cdot \vec{a}_1 + 2\vec{a}_2 = 1\begin{bmatrix} 2 \\ 1 \end{bmatrix} + 2\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} + \begin{bmatrix} -2 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \end{bmatrix} = \vec{b}.$$

Fact: $\boxed{!}$ In general, the solution set of the vector equation

$$x_1\vec{a}_1 + x_2\vec{a}_2 + \cdots + x_n\vec{a}_n = \vec{b}$$

is the same as the solution set of the linear system with augmented matrix:

$$\begin{bmatrix} \vec{a}_1 & \vec{a}_2 & \cdots & \vec{a}_n & \vec{b} \end{bmatrix}$$

where each vector is a column in the larger matrix.

Notice, $\vec{b} \in \text{span}\{\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_n\}$ if this equation (and linear system) has a solution.

Matrix equations: $A\vec{x} = \vec{b}$

Later we will see matrix multiplication more fully, but for now, we focus on a pertinent special case.

Def: The **product** of an $m \times n$ matrix A and a vector $\vec{x} \in \mathbb{R}^n$, denoted $A\vec{x}$, is the linear combination of the n columns of A with weights the n-entries of $\vec{x}$.

$$A\vec{x} = \begin{bmatrix} \vec{a}_1 & \vec{a}_2 & \cdots & \vec{a}_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = x_1\vec{a}_1 + x_2\vec{a}_2 + \cdots + x_n\vec{a}_n$$

Ex:

$$\begin{bmatrix} 1 & 3 & 0 \\ 2 & 4 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} = 1\begin{bmatrix} 1 \\ 2 \end{bmatrix} + 0\begin{bmatrix} 3 \\ 4 \end{bmatrix} + 2\begin{bmatrix} 0 \\ 5 \end{bmatrix} = \begin{bmatrix} 1 \\ 12 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 7 \\ 3 & 9 \\ 5 & 11 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix} - \begin{bmatrix} 7 \\ 9 \\ 11 \end{bmatrix} = \begin{bmatrix} -6 \\ -6 \\ -6 \end{bmatrix} = -\begin{bmatrix} 6 \\ 6 \\ 6 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 7 \\ 3 & 9 \\ 5 & 11 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix} - \begin{bmatrix} 7 \\ 9 \\ 11 \end{bmatrix} = \begin{bmatrix} -6 \\ -6 \\ -6 \end{bmatrix} = -\begin{bmatrix} 6 \\ 6 \\ 6 \end{bmatrix}$$

$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= 1 \\ 5x_1 - x_2 &= -1 \end{aligned} \implies x_1\begin{bmatrix} 1 \\ 5 \end{bmatrix} + x_2\begin{bmatrix} 2 \\ -1 \end{bmatrix} + x_3\begin{bmatrix} 3 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\underbrace{\begin{bmatrix} 1 & 2 & 3 & 1 \\ 5 & -1 & 0 & -1 \end{bmatrix}}_{\text{coeff. matrix}} \implies \underbrace{\begin{bmatrix} 1 & 2 & 3 \\ 5 & -1 & 0 \end{bmatrix}}_{A}\underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{\vec{x}} = \underbrace{\begin{bmatrix} 1 \\ -1 \end{bmatrix}}_{\vec{b}}$$

"So, by the definition of matrix multiplication, a linear system can be written as the coefficient matrix A times the vector of variables $\vec{x}$ set equal to the right hand side, as a vector, $\vec{b}$."

To continue this analogy, the question of solving a system is restated in the following fact:

<u>Fact</u>: The equation $A\vec{x} = \vec{b}$ has a solution (a choice of $\vec{x}$) if and only if $\vec{b}$ is a linear combination of the columns of A.

<u>Ex</u> Is every $\vec{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$ a linear combination of the columns of

$$A = \begin{bmatrix} 1 & 3 & 4 \\ -4 & 2 & -6 \\ 0 & 14 & 10 \end{bmatrix}?$$

Equivalently, does $A\vec{x} = \vec{b}$ have a solution for every $\vec{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$.

Note: $A\vec{x} = \vec{b} \implies \begin{bmatrix} 1 & 3 & 4 \\ -4 & 2 & -6 \\ 0 & 14 & 10 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$

$$\implies \begin{bmatrix} 1 & 3 & 4 & b_1 \\ -4 & 2 & -6 & b_2 \\ 0 & 14 & 10 & b_3 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 4 & b_1 \\ 0 & 14 & 10 & b_2 + 4b_1 \\ 0 & 14 & 10 & b_3 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 3 & 4 & b_1 \\ 0 & 14 & 10 & b_2 + 4b_1 \\ 0 & 0 & 0 & b_3 - (b_2 + 4b_1) \end{bmatrix}$$

The last row $0 = b_3 - (b_2 + 4b_1)$

$b_1 = 0$ 0

The last row $0 = b_3 - (b_2 + 4b_1)$ is contradictory for many choices of b_1, b_2, b_3 say $\begin{matrix} b_1 = 0 \\ b_2 = 0 \\ b_3 = 1 \end{matrix}$ or $\begin{matrix} 0 \\ 1 \\ 2 \end{matrix}$

So no, in particular $\vec{b} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ is <u>not</u> a lin. comb. of the columns of A

"Note, if A had had a third pivot $\begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix}$ then this wouldn't have worked, that is, every $\vec{b}$ <u>would</u> have been a lin. comb. of the columns of A."

Thus a study of the coefficient matrix A leads us to answer the harder question "is every $\vec{b}$ a linear comb. of the columns of A?"

Notice, every $\vec{b}$ is just every vector in $\mathbb{R}^n$ so we obtain, this equivalence

For an $m \times n$ matrix A, the following are equivalent:

1) For each $\vec{b}$ in $\mathbb{R}^n$, the equation $A\vec{x} = \vec{b}$ has a solution
2) Every $\vec{b}$ is a linear comb. of the columns of A.
3) The columns of A span $\mathbb{R}^n$
4) A has a pivot position in every row.

To conclude: a "more efficient" way of computing matrix products:

The <u>Row-Vector Rule</u> for $A\vec{x}$:

The ith entry of $A\vec{x}$ is the sum of the products of corresponding entries of the ith row of A with $\vec{x}$.

Using this we can quickly verify:

1) $A(\vec{u} + \vec{v}) = A\vec{u} + A\vec{v}$
2) $A(c\vec{u}) = c(A\vec{u})$

<u>Ex</u> Compute $A\vec{x}$ if $A = \begin{bmatrix} 2 & 3 & 2 \\ 1 & 0 & 1 \\ 5 & 6 & 7 \end{bmatrix}$ and $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$.

Ex] Compute $A\vec{x}$ if $A = \begin{bmatrix} 2 & 3 & 2 \\ 1 & 0 & 1 \\ 5 & 6 & 7 \end{bmatrix}$ and $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$.

$$A\vec{x} = \begin{bmatrix} 2 & 3 & 2 \\ 1 & 0 & 1 \\ 5 & 6 & 7 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$= \begin{bmatrix} 2x_1 + 3x_2 + 2x_3 \\ x_1 + x_3 \\ 5x_1 + 6x_2 + 7x_3 \end{bmatrix} = x_1\begin{bmatrix} 2 \\ 1 \\ 5 \end{bmatrix} + x_2\begin{bmatrix} 3 \\ 0 \\ 6 \end{bmatrix} + x_3\begin{bmatrix} 2 \\ 1 \\ 7 \end{bmatrix}$$

"Once we recognize the common way this linear combination simplifies we become much faster at these matrix-vector products."